Direct Research Journals

LIGHTLIKE PARALLEL VECTOR FIELDS AND EINSTEIN'S EQUATIONS OF GRAVITY: A PARTIAL SOLUTION

Direct Research Journal of Engineering and Information Technology Vol.3 (1), pp. 1-6, February, 2015
Available online at directresearchpublisher.org/drjet
ISSN 2354-4155

Research Paper

Muremyi Roger

I.C.K (Catholic Institute of Kabgayi), in Rwanda.

ABSTRACT

The purpose of this paper was to find out the lightlike parallel vector field and Einstein's equation of gravity, specifically a partial solution. The pseudo –Riemannian manifold and gravitational field through Einstein's equation have been used , the curvature tensor in 4 dimensional space and the Christoffel symbols in 3-dimensional are equal to Christoffel symbols in 4-dimensional space by using contraction properties of the tensors and as the results the existence of lightlike parallel vector field implies that the space time is flat, this leads us to non existence of the lightlike parallel vector field in a non trivial gravitational field.

Key words: Tensors, Parallel vector field, differential manifold, smooth manifold, Riemannian connection, Christoffel symbols.

*Corresponding Author E-mail: muremyiroger@yahoo.fr
+250788793493

Accepted 23 December, 2014

INTRODUCTION

The present work deals with the existence of lightlike parallel vector fields on Riemannian manifolds describing gravitational fields. In Euclidian geometry , we know what means parallel displacement . the concept of affine connection allows the definition of an infinitesimal parallel displacement of a vector at a point p of a manifold M. but generally, the process is not integrable, this meaning that parallel displacement of a vector $\overrightarrow{v_p}$ at a point p along two different curves joining p to the same point p' does not yield necessary the same vector $\overrightarrow{v_p}$ at p'. If there exists at p some vector $\overrightarrow{v_p}$, whose the displacement vector at any point $p' \in M$.does not depend on the particular smooth curve joining p to p', the process yields a vector field on the manifold M which we call a parallel vector field. Through previous research on the subject the existence of nontrivial 4 dimensional Riemannian manifolds, admitting parallel vector fields are easily proved. The present paper investigates the existence of lightlike parallel vector fields in a gravitational field governed by Einstein's equations of general relativity.

Elements of differential geometry

Differential geometry is a mathematical discipline that uses the methods of differential and integral calculus to study problem in geometry. In mathematics specifically differential geometry, the infinitesimal geometry concerned more generally with geometric structure on differential take manifold, it is closely related with differential topology and with the geometric aspects of the theory of differential equation (Mishechenko et al., 1980).

Vector field on a differentiable manifold

A fundamental ingredient in formulating, the notion of differential manifolds is that of homeomorphism.

Homeomorphism

Let X and Y be two topological spaces and let h be a map: h$\rightarrow X \rightarrow Y$. If h is one- to –one and if both h and its inverse h are continuous, then, we say that h is a homeomorphism (Mishechenko et al., 1980).

A topological manifold

A topological manifold is a second countable, Hausdorff space which is locally homeomorphism to Euclidean space, by a collection (called Atlas) of homeomorphisms called chart. The composition of one chart with the inverse of another chart is a function called a transition map, and defines a homeomorphisms of an open subset of Euclidean space onto another open subset of Euclidean space (Mishechenko et al., 1980).

Atlas and open charts

In generally to know a differentiable manifold, we have to know the intuitive idea of an atlas of open charts suitable reformulated in mathematical terms, provide the very definition of a differentiable manifold.

Atlas

An atlas on a topological space X is a collection of pairs {(U_α, φ_α)} called charts, where the U_α are open sets which cover X, and for each index $\alpha, \varphi_\alpha : U_\alpha \to \mathbb{R}^n$ is homeomorphism of U_α onto an open subset of n-dimensional Euclidean space. The transition maps of the atlas are the functions:

$$\phi_{\alpha\beta} = \phi_\alpha o \phi_\beta^{-1} / \phi_\beta(U_\alpha \cap U_\alpha) \to \phi_\beta(U_\alpha \cap U_\alpha) (\text{do Carmo,Manfredo Perdigao,1994}).$$

Differentiable structure

Let M be a topological hausdorff space, A differentiable structure of dimension on M is an atlas X= $u_\alpha \in (u_\alpha, u_\alpha)$ open

charts(u_α, u_β),

where $\alpha \in x,\ u_\alpha \in M$ (Kreyszig, Erwin, 1991) .

An open charts

Let now M be a topological space, an open charts of M is a pair (U,φ), where U $\subset M$ is an open subset of M and φ is a homeomorphism of U on an open subset $\mathbb{R}^n$ (do Carmo, Manfredo Perdigao, 1994).

Differentiable manifold

Differentiable manifold is a topological manifold equipped with an atlas whose transition maps are all differentiable. More generally a c^k- manifold is a topological manifold with an atlas whose transition maps are all k-times continuously differentiable (do Carmo, Manfredo Perdigao 1994)

A smooth manifold

A smooth manifold or c^∞- manifold is a differentiable for which all the transitions maps are smooth. That is the derivatives of all orders exist, so it is a c^k manifold for all k.

Covariant differentiation and parallel displacement

In mathematics, the covariant derivative is a way of specifying a derivative along tangent vectors of a manifold. Alternatively , the covariant derivative is a way of introducing and working with a connection with the approach given by a principal connection on the frame bundle.

Equation of parallel displacement

Comparison of tangent vectors at different points on a manifold is generally not a well defined process. An affine connection provides one way to remedy this using the notion of parallel transport, and indeed this can be used to give a definition of an affine connection. Let M be a manifold with an affine connection $\nabla,$ then a vector field X is said to be parallel if $\nabla X = 0$, in the sense that for any vector field Y, $\nabla Y = 0,$ intuitively speaking parallel vectors have all their derivatives equal to zero and are therefore in some sense constant.
The parallel displacement is performed a long curve $\gamma : [a, b] \to M^n$.
Let $p \in M^n and\ Q \in M^n, where\ M^n$ is a smooth manifold p and Q are connected by a curve γ. Let us assume that $\gamma \in M^n$, is provided with an affine connection $\nabla = \{\nabla_k\}$ where o$\leq k \leq 1$, covariant derivative of a tensor field $T = T^\alpha_\beta$ a long γ, $\nabla_j = \xi^k \nabla_k$, we say that the tensor field T is parallel along the curve $\gamma\ if\ \nabla_j(T) \equiv 0$, along γ. Let $\gamma(t) \subseteq M^n$ be a smooth

curve and let a field T=$\{T^i\}$ be given along this curve . This is said to be parallel along $\gamma(t)$ relative to the connection $\nabla_{\gamma(t)} = 0.$

The equation $\frac{dT^i}{dt} + \Gamma^i_{pk}\frac{dx^k}{dt}T^p = 0$, is called the equation of parallel displacement along a curve $\gamma(t),$ for different curves γ we can obtain different equations of parallel displacement.

Covariant differentiation

Covariant ∇ is said to be defined on a smooth manifold M^n if for each smooth atlas there is a given in each chart a set of smooth functions

$$\Gamma'^i_{\alpha\beta} = \sum_{i,j,k} \frac{\partial x_i}{\partial y_\alpha}\frac{\partial x_j}{\partial y_\beta}\frac{\partial x_\gamma}{\partial y_k}\Gamma^k_{ij} + \sum_j \frac{\partial^2 x_j}{\partial y_\alpha}\frac{\partial x_j}{\partial y_\beta}\frac{\partial y_\gamma}{\partial y_j}, \qquad (1)$$

then ∇ is given by the formula

$$(\nabla T)^{i_1,\dots,i_k}_{j_1,\dots,j_p,\alpha} = \frac{\partial}{\partial x^\alpha}(T)^{i_1,\dots,i_k=q}_{j_1,\dots,j_p,} + \sum_{s=1}^{k}(T)^{i_1,\dots,i_s=q}_{j_1,\dots,j_p,} - \sum_{s=1}^{p}(T)^{i_1,\dots,i_k}_{j_1,\dots,j_s=q,\dots,j_p}\Gamma^q_{js^\alpha} \qquad (2),$$

covariant differentiation intuitively , by a parallel vector field, we mean a vector field with the propert that the vectors at different points are parallel.

Riemannian Manifold

In Riemannian geometry, a Riemannian manifold (M, g) with Riemannian metric g is real differentiable manifold M in which each tangent space is equipped with inner product g in a manner which varies smoothly from point to point. The metric g is a positive definite metric tensor.

Tensor analysis on a Riemannian manifold

In the mathematical field of differential geometry, a metric tensor is a type of function defined on a manifold which takes as input a pair of tangent vectors v and w and produces a real number g(v,w) in a way that generalizes many of the familiar properties of the dot product of vectors in Euclidean space. Tensor analysis is very important for mathematicians and physics (Theodore, 2004).

The metric tensor

The metric tensor is a central object in general relativity that describes the local geometry of space time (as a result of solving the Einstein field equation). The metric is a symmetric tensor and is an important mathematical tool. As well as being used to raise and lower tensor indices, it also generates the connections which are used to construct the geodesic equations of motion and the Riemann curvature tensor.

The Riemann curvature tensor

A useful way of measuring the curvature of a manifold is with an object called the Riemann curvature Tensor. This tensor measures curvature by use of an affine connection by considering the effect of parallel transporting a vector between two points along two curves. The discrepancy between the results of these two parallel transport routes is essentially quantified by the Riemann tensor. The curvature of Riemannian manifold can be described in various ways , the most standard one is the curvature tensor given in terms of a LEVI-CIVITA connection or covariant differentiation ∇ and Lie bracket [*,*] by the following formula :

R(u, v)w=$\nabla_u\nabla_v w - \nabla_v\nabla_v w - \nabla_{[u,v]}w$, here R(u,v) is a linear transformation of the tangent space of the manifold, it is linear in each argument. If $u = \frac{\partial}{\partial x_i}$, and $v = \frac{\partial}{\partial x_j}$ are coordinate vector fields. Then $\nabla_{[u,v]}$=0 and therefore , the formula simplifies to

R(u,v)w=$\nabla_u\nabla_v w - \nabla_v\nabla_v w$ (Alfred, 1998).

Symmetries and identities

The curvature tensor has the following symmetries

1. $\mathrm{R(u,v)} = -\mathrm{R(u,v)}$
2. $\langle \mathrm{R(u,v)w,z} \rangle = \langle -\mathrm{R(u,v)z,w} \rangle$, where $\langle\ \rangle$ are scalar product
3. $R(u,v)w + R(u,w)u + R(w,u)v$=0

The last identity was discovered by Ricci, but is often called the first Bianchi identity, just because it looks similar to Bianchi identity. Let (M, g) be a Riemannian manifold, then an affine connection ∇ is called a Levi-Civita connection if : it preserves the metric for any vector field X,Y,Z we have X(g(Y,Z)$)$ $\equiv$g($\nabla_x Y, Z$)+ g($Y, \nabla_x Z$) where X(g(Y,Z)$)$denotes the derivative of the function g(Y,Z)along the vector field X.

The Levi- Civita Connection

If (M, g) is a Riemannian manifold then there is a unique affine connection ∇ on M with the following properties: the connection is torsion free ∇ is zero; the second condition means that the connection is a metric connection in the sense that the Riemannian metric g is parallel : ∇g=0. In local coordinates the components of the connection form are called Christoffel symbols because of uniqueness of the Levi-Civita Connection, there is a formula for these components in terms of the components of g (Kreyszig, 1991).

Properties of curvature tensor

First the curvature tensor is antisymmetric on index 1 and m$R^i_{klm} = -R^i_{klm}$, (3)

we verify the identities $R^i_{klm} + R^i_{mkl} + R^i_{lmk} = 0$, the mixt tensor R^i_{klm} is used in covariant components $R^i_{iklm} = g_{in} R^n_{klm}$ (4)

By simple transformations we obtain

$$R^i_{iklm} = \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{km}}{\partial x^i \partial x^l} + g_{np}\left(\Gamma^n_{kl}\Gamma^p_{im} - \Gamma^n_{km}\Gamma^p_{il}\right)\right) \quad (5)$$

In calculations the last term of (5) it can be written as

$g^{np}(\Gamma_{n,kl}\Gamma_{p,im} - \Gamma_{n,km}\Gamma_{p,il})$ that why we have the following properties :

$\mathrm{R}_{iklm} = -\mathrm{R}_{kilm} = -\mathrm{R}_{iklm}$, $\mathrm{R}_{iklm} = \mathrm{R}_{lmik}$, we define the Ricci Tensor as follows: $\mathrm{R}_{ik} = g^{lm}\mathrm{R}_{limk} = R^l_{ikl}$, it is easy to check that

$$\mathrm{R}_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{kl} \quad (6)$$

Where $\Gamma^i_{jk} = \frac{g^{i\alpha}}{2}\left(\frac{\partial g_{k\alpha}}{\partial x^j} + \frac{\partial g_{j\alpha}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^\alpha}\right.$ (7)

The Γ^i_{jk} are called the Christoffel symbols,

this tensor is symmetric: $\mathrm{R}_{ik} = \mathrm{R}_{ki}$, hence we obtain in contracting R_{ik} , we obtain

R= $\mathrm{g}^{ik}\mathrm{R}_{ik}$= $\mathrm{g}^{il}\mathrm{g}^{km}\mathrm{R}_{iklm}$ (8)

which is the scalar curvature of the manifolds.

Contravariant vector and covariant vector

Contravariant vector t is the vector of quantities T_i in which its coordinates change as the differential likes

$$T_i = \frac{\partial x^i}{\partial x'^k} T'^k$$

Covariant vector is the vector of quantities T_i , which are transformed in the coordinates change like the derivative of scalar and it is written $T_i = \frac{\partial x^i}{\partial x'^k} T'_k$.

Einstein's equations of Gravity

The basis of Einstein's general relativity is the idea that the space the space time is a pseudo- Riemannian manifold whose metrics tensor itself is a dynamical object. It can be shown from the principal of least action that Einstein's equation of gravity read

$$R_{ij} - \frac{1}{2} R R_{ij} = K T_{ij} \quad (9)$$

where R_{ij} is the Ricci Tensor , $T_{ij}$is Energy momentum tensor , K is Einstein constant gravitation, R is a scalar curvature tensor.

Gravitation fields in classical physics

At the base of the classical physics is the notion that a body's motion can be described as a combination of free motion, and deviations from its free motion such deviation are caused by external forces acting on a body in coordance with Newton's second law of motion which state that the net forces acting on a body is equal to that body's mass multiplied by its acceleration. According to Newton's law of gravity and independent verified by experiments there is a universality of free fall, known as weak equivalence principle or the universal equality of inertial and passive gravitational mass. The trajectory of a test body in free fall dependents only on its position and inertial speed but not on any of its material properties.

Gravitational fields in relativistic physics

According to general relativity , the observed gravitational attraction between masses results from the warping of space and time those masses. General relativity or the general theory of relativity is the geometric theory of gravitation published by Einstein. It is the current description of gravitation in modern physics. It unifies special relativity and Newton's law of universal gravitation and describes gravity as a geometry property of space and time , or space-time (Theodore, 2004).

Lightlike Parallel vector fields in gravitational fields

Parallelism

The vector field X is said to be parallel if it is parallel along any smooth curve, this implies that if its covariant derivative are equal to zero in any coordinate system. Consider

$$\frac{\partial x^i}{\partial x^k} + \Gamma^j_{jk} X^j = 0 \Leftrightarrow \frac{\partial x^i}{\partial x^k} = -\Gamma^j_{jk} X^j \quad (10)$$

$$\frac{\partial X^i}{\partial x^l} + \Gamma^i_{jk} X^j = 0 \Leftrightarrow \frac{\partial X^i}{\partial x^l} = -\Gamma^i_{jk} X^j \quad (11)$$

$$\frac{\partial}{\partial x^l}\left(\frac{\partial X^i}{\partial x^k}\right) = -\frac{\partial}{\partial x^l}\left(\Gamma^i_{jk} X^j\right) \quad (12)$$

$$\frac{\partial}{\partial x^k}\left(\frac{\partial X^i}{\partial x^l}\right) = -\frac{\partial}{\partial x^k}\left(\Gamma^i_{jk} X^j\right) \quad (13)$$

$$\frac{\partial}{\partial x^k}\left(\frac{\partial X^i}{\partial x^l}\right) - \frac{\partial}{\partial x^l}\left(\frac{\partial X^i}{\partial x^k}\right) = 0 \quad (14)$$

$$\frac{\partial}{\partial x^k}\left(\Gamma^i_{jl} X^j\right) - \frac{\partial}{\partial x^l}\left(\Gamma^i_{jk} X^j\right) = 0 \quad (15)$$

$$\frac{\partial \Gamma^i_{jl}}{\partial x^k}\left(X^j\right) + \Gamma^i_{jl}\frac{\partial X^j}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^k}\left(X^j\right) - \Gamma^i_{jk}\frac{\partial X^j}{\partial x^l} = 0 \quad (16)$$

The equation (16) can be written as

$$\left(\frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^k}\right) X^j + \Gamma^i_{jl}\frac{\partial x^j}{\partial x^k} - \Gamma^i_{jk}\frac{\partial x^j}{\partial x^l} = 0 \quad (17)$$

$$\frac{\partial X^j}{\partial x^k} + \Gamma^j_{km} X^m = 0, \Leftrightarrow \frac{\partial X^j}{\partial x^k} = -\Gamma^j_{km} X^m, \quad (18)$$

$$\frac{\partial x^j}{\partial x^l} + \Gamma^j_{lm} X^m = 0, \Leftrightarrow \frac{\partial x^j}{\partial x^l} = -\Gamma^j_{lm} X^m \quad ,(19)$$

, replacing (18) and (19) into (17) we obtain

$$\left(\frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^k}\right) X^j + \Gamma^i_{jl}\left(-\Gamma^j_{km} X^m\right) - \Gamma^i_{jk}\left(-\Gamma^j_{lm} X^m\right) = 0 \quad (20),$$ *which is* equivalent to

$$\left(\frac{\partial \Gamma^i_{ml}}{\partial x^k} - \frac{\partial \Gamma^i_{km}}{\partial x^l} + \Gamma^i_{jk}\Gamma^j_{lm} - \Gamma^i_{jl}\Gamma^i_{mk}\right) X^m = 0 \quad (21),$$

this equation can be written as $R^i_{lkm} X^m = 0$, where R^i_{lkm} is the Riemanian curvature tensor. As the consequence we have $R_{ij} X^j = 0$, with R_{ij} the Ricci Tensor.

Special case

Let us assume that X is a lightlike parallel vector field, then g(X,X)=0, since X is a regular at any point, we can take $X = \frac{\partial}{\partial x^0}$ and we will get $g(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^0}) \equiv g_{oo} = 0$, it will be easy to show that $\frac{\partial g_{ij}}{\partial x^0} = 0$. Let us consider the particular case where $g^{00} \neq 0$, then we can defined the 3 dimensional metric as $\gamma^{\alpha\beta} = g^{\alpha\beta} - \frac{g^{0\alpha} g^{0\beta}}{g^{00}}$. Let us consider the Einstein's equations $R_{ij} - \frac{1}{2} R g_{ij} = \kappa T_{ij}$ (22), where $T_{ij} = (p+\varepsilon) u_i u_j - p g_{ij}$ for macroscopic bodies, with ε is energy density.

Hence

$$g^{ij} R_{ij} - \frac{1}{2} R g_{ij} g^{ij} = \kappa\left[(p+\varepsilon) u_i u_j g^{ij} - p g_{ij} g^{ij}\right]$$

and , since we have $g_{ij} g^{ij} = 4$ and $u_i u^i = 1$, we get R-2R= $\kappa\left[(p+\varepsilon) u_i u_j - 4p\right] = \kappa\left[(p+\varepsilon) - 4p\right]$ with R= $g^{ij} R_{ij}$, Ttherefore, R= $\kappa\left[(3p - \varepsilon)\right.$. We know that κ is a constant, $\kappa \neq 0, -3p - \varepsilon = 0$ and since $\varepsilon \geq 0$, $p \geq 0$, we get $\varepsilon = 0, p = 0$, and of course R=0. Thus $R_{ij} = 0$. The curvature tensor for the 3 dimensional space is given by

$$P_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\frac{\partial^2 \gamma_{\alpha\delta}}{\partial x^\beta \partial x^\gamma} + \frac{\partial^2 \gamma_{\beta\gamma}}{\partial x^\alpha \partial x^\delta} - \frac{\partial^2 \gamma_{\alpha\gamma}}{\partial x^\beta \partial x^\delta} - \frac{\partial^2 \gamma_{\beta\delta}}{\partial x^\alpha \partial x^\gamma}\right) + \gamma_{np}\left(\Lambda^n_{\alpha\delta}\Lambda^p_{\beta\gamma} - \Lambda^n_{\alpha\gamma}\Lambda^p_{\beta\delta}\right)$$
(23)

and since $\gamma_{\alpha\gamma} = g_{\beta\gamma}$,

this tensor becomes

$$P_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\frac{\partial^2 g_{\alpha\delta}}{\partial x^\beta \partial x^\gamma} + \frac{\partial^2 g_{\beta\gamma}}{\partial x^\alpha \partial x^\delta} - \frac{\partial^2 g_{\alpha\gamma}}{\partial x^\beta \partial x^\delta} - \frac{\partial^2 g_{\beta\delta}}{\partial x^\alpha \partial x^\gamma}\right) + g_{np}\left(\Lambda^n_{\alpha\delta}\Lambda^p_{\beta\gamma} - \Lambda^n_{\alpha\gamma}\Lambda^p_{\beta\delta}\right)$$

This tensor will be equal to the curvature tensor in 4 dimensional space iff the Christoffel symbols in 3-dimensional are equal to Christoffel symbols in 4-dimensional space

$$\Lambda^\alpha_{\beta\gamma} = \frac{1}{2}\gamma^{\alpha\lambda}\left(\frac{\partial g_{\beta\lambda}}{\partial x^\gamma} + \frac{\partial g_{\gamma\lambda}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\lambda}\right) = \frac{1}{2}\left(g^{\alpha\lambda} - \frac{g^{0\alpha} g^{0\lambda}}{g^{00}}\right)\left(\frac{\partial g_{\beta\lambda}}{\partial x^\gamma} + \frac{\partial g_{\gamma\lambda}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\lambda}\right) \quad (24)$$

$$\Lambda^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda}\left(\frac{\partial g_{\beta\lambda}}{\partial x^\gamma} + \frac{\partial g_{\gamma\lambda}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\lambda}\right) - \frac{1}{2}\frac{g^{0\alpha} g^{0\lambda}}{g^{00}}\left(\frac{\partial g_{\beta\lambda}}{\partial x^\gamma} + \frac{\partial g_{\gamma\lambda}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\lambda}\right)$$

$$\Lambda^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma}\frac{1}{2}\frac{g^{0\alpha} g^{0\lambda}}{g^{00}}\left(\frac{\partial g_{\beta\lambda}}{\partial x^\gamma} + \frac{\partial g_{\gamma\lambda}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\lambda}\right) \quad (25)$$

$\Lambda^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \frac{g^{0\alpha}}{g^{00}}\Gamma^0_{\beta\gamma}$, as we have $\frac{1}{g^{00}} = g_{00}$, $\frac{g^{0\alpha}}{g^{00}} = g_{00} g^{0\alpha} = \delta^\alpha_0 = 0$.

$\Lambda^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \delta^\alpha_0 \Gamma^0_{\beta\gamma}$, and $\delta^\alpha_0 = 0$, for $(\alpha = 1,2,3)$ (26), which implies that $\Lambda^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma}$. Thus the equation (23) is equal to $R_{\alpha\beta\gamma\delta}$, where

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\frac{\partial^2 g_{\alpha\delta}}{\partial x^\beta \partial x^\gamma} + \frac{\partial^2 g_{\beta\gamma}}{\partial x^\alpha \partial x^\delta} - \frac{\partial^2 g_{\alpha\gamma}}{\partial x^\beta \partial x^\delta} - \frac{\partial^2 g_{\beta\delta}}{\partial x^\alpha \partial x^\gamma}\right) + g_{np}\left(\Gamma^n_{\beta\gamma}\Gamma^p_{\alpha\gamma} - \Gamma^n_{\alpha\beta}\Gamma^p_{\gamma\delta}\right) \quad (27)$$

Since $\gamma_{\alpha\gamma} = g_{\beta\gamma}$, thus $P_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}$, we find that the curvature $P_{\alpha\beta\gamma\delta}$ Corresponding to $\gamma_{\alpha\gamma}$ is equal to the curvature $R_{\alpha\beta\gamma\delta}$ Corresponding to g_{ij}, which is the metric tensor for the four dimensional space.

The Ricci Tensor in three dimensional space

$P_{\alpha\beta=}\gamma^{\gamma\delta}\, P_{\gamma\alpha\delta\beta}$ and $g_{\gamma\lambda}\gamma^{\gamma\delta} = \delta^{\delta}_{\lambda}$ multiply $P_{\alpha\beta=}\gamma^{\gamma\delta}\, P_{\gamma\alpha\delta\beta}$ by $g_{\gamma\lambda}$ then we obtain $g_{\gamma\lambda}$ $P_{\alpha\beta=}\, g_{\gamma\lambda}\gamma^{\gamma\delta} P_{\gamma\alpha\delta\beta} = \delta^{\delta}_{\lambda}$,

$g_{\gamma\delta}g^{\gamma\tau}P_{\alpha\beta} = g^{\gamma\tau}R$, for R=P, $\delta^{\tau}_{\delta}P_{\alpha\beta} = g^{\gamma\tau}R$, $\delta^{\tau}_{\delta} = \begin{cases} 1, if\ \tau = \delta \\ 0, if\ \tau \neq \delta \end{cases}$,

$P_{\alpha\beta=}\ g^{\gamma\delta} P_{\gamma\alpha\delta\beta} R_{\alpha\beta}$,

The Ricci tensor in three dimensional space is equal to the Ricci tensor in four dimensional space, from the contraction of the Einstein's equation of gravity R=0 which implies that $R_{\alpha\beta} = 0 \Longrightarrow P_{\alpha\beta} = 0 \Longrightarrow P_{\gamma\alpha\delta\beta} = 0$ and since $P_{\gamma\alpha\delta\beta} = R_{\gamma\alpha\delta\beta}$,then $R_{\alpha\beta\gamma\delta} = 0.$

Concluding remark

It has been shown that, in pseudo –Riemannian manifold describing a gravitational field through Einstein's equation, the existence of lightlike parallel vector field implies that the space time is flat, this meaning that there does not exist any lightlike parallel vector field in a non trivial gravitational field.

REFERENCES

Alfred G (1998). *Modern Differential Geometry of Curves and Surfaces/with Mathematica* (2nd ed.).

do Carmo, Manfredo Perdigao (1994). *Riemannian Geometry*.

Mishechenko A, and et al. 1980), course of differential Geometry and topology, Edition Mir Moskow.

Theodore F (2004). *The geometry of physics: an introduction* (2nd ed.).

Kreyszig E (1991). *Differential Geometry*. Good classical geometric approach to differential geometry with tensor machinery.