Direct Research Journal of Engineering and Information Technology (DRJEIT) Vol.2 (2), pp. 11-17, May 2014
ISSN 2354-4155 Available online at directresearchpublisher.org/drjet

Bounds on cost measures in terms of 'useful' R-norm information measures

Research Paper

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Accepted 28 April 2014

In the present communication a new generalized cost measure is defined and its lower and upper bounds in terms of 'Useful' R-norm information measures are obtained. Further, a generalized cost measure of degree β is introduced by replacing probability distribution P by probability distribution of degree and its bounds have been studied in terms of 'useful' R-norm information measure of degree β. Particular cases have also been studied.

Mathematics Subject Classification: 94A17 and 94A24.

Key words and Phrases: Mean code words length; Kraft's inequality; Code Alphabets; R-norm entropy and Holder's inequality.

INTRODUCTION

Let us consider the set of positive real numbers, not equal to 1 and denote this by R^+ defined as $R^+ = \{R : R > 0, R \neq 1\}$. Let Δ_n with $n \geq 2$ is the set of all probability distributions

$$P = \left\{ (p_1, p_2, \ldots, p_n), \; p_i \geq 0, \text{ for each i and } \sum_{i=1}^{n} p_i = 1 \right\}.$$

Boekee and Lubbe, (1980) studied R-norm information of the distribution P defined for $R \in R^+$ by

$$H_R(P) = \frac{R}{R-1}\left[1 - \left(\sum_{i=1}^{n} p_i^R\right)^{1/R}\right]. \tag{1}$$

The R-norm information measure (1) is a real function $\Delta_n \to R^+$ defined on Δ_n, where $n \geq 2$ and R^+ is the set of real positive numbers. The measure (1) is different from entropies of Shannon (1948), Renyi (1976), Havrda and Charvat (1967) and Daroczy (1970). The main property of this measure is that when $R \to 1$, (1) approaches to Shannon's entropy and when $R \to \infty$, $H_R(P) \to (1 - \max p_i)$, where $i = 1, 2, \ldots, n.$

The measure (1) can be generalized in so many ways. Hooda et al. (2013) proposed and characterized the following generalization of (1):

$$H_R(P;U) = \frac{R}{R-1}\left[1 - \left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}}\right]. \tag{2}$$

where $U = (u_1, u_2, u_3, \ldots, u_n)$ is the utility distribution and $u_i > 0$ is the utility of an event with probability p_i. It should be noted that if $R \to 1$ then (2) reduced to

$$H(P;U) = -\frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{i=1}^{n} u_i p_i} \tag{3}$$

(3) is the 'useful' information measure due to Longo, (1972),which was also characterized by Bhaker and Hooda (1993) by mean value representation. We call the measure (2) as 'useful' R-norm information measure.

In the present paper we give a brief account of some mean codeword lengths as the generalized cost measure in section 2 and study their lower and upper bounds of generalized cost measure in terms of generalized 'useful' R-norm information measure in section 3. In section 4 we derive the lower and upper bounds of generalized cost measure of degree β in terms of generalized 'useful' R-norm information measure of degree β.

The Generalized Cost measure

Let a finite set of n source symbols $X=\{x_1, x_2, \ldots, x_n\}$ with probabilities $P=\{p_1, p_2, \ldots, p_n\}$ be encoded using $D \geq 2$ code alphabets, then there is a uniquely decipherable instantaneous code with lengths $l_1, l_2, \ldots, l_n$ if and only if

$$\sum_{i=1}^{n} D^{-l_i} \leq 1 \tag{4}$$

which is known as Kraft's inequality given by Kraft (1949).

Let $$L=\sum_{i=1}^{n} p_i l_i \log D \tag{5}$$

be the average codeword length associated with input symbols $X=(x_1, x_2, \ldots, x_n)$ under the Kraft's inequality Shannon (1948) proved the following result for a noiseless channel:

$$H(P) \leq L < H(P) + \log D, \tag{6}$$

with equality if and only if $l_i = -\log_D p_i$ for $i=1,2,\ldots,n$.

Guiasu and Picard (1971) considered the problem of encoding by means of a single letter prefix code whose code words $w_1, w_2, \ldots, w_n$ are of lengths $l_1, l_2, \ldots, l_n$ respectively and satisfy the Kraft's inequality (1949).They introduced the following 'useful' mean length of code:

$$L(P;U)=\frac{\sum_{i=1}^{n} u_i p_i l_i}{\sum_{i=1}^{n} u_i p_i}. \tag{7}$$

Later on Longo,(1972) interpreted (7) as the average transmission cost of the letters x_i's and obtained the following bounds for the cost measure (7):

$$H(P;U) \leq L(P;U) < H(P;U) + 1, \tag{8}$$

where $H(P;U)$ given by (3).

It may be noted that the mean codeword length (7) had been generalized parametrically by many authors and their bounds had been studied in terms of generalized measures of entropies. Here we give the following new generalization of (7) and study its bounds in terms of 'useful' R-norm information measure given by (2)

$$L_R(P;U)=\frac{R}{R-1}\left(1-\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} u_i p_i}\right), \tag{9}$$

If $u_i = 1$ for all $i=1,2,\ldots,n$, (9) reduces to

$$L_R(P)=\frac{R}{R-1}\left(1-\sum_{i=1}^{n} p_i D^{-l_i\left(\frac{R-1}{R}\right)}\right), \tag{10}$$

which is average codeword length due to Boekee and Lubbe (1980). In case $R \to 1$, then (9) reduces to (7) and after ignoring utilities it reduces to (5).

A Coding Theorem on Bounds

In this section we study the lower and upper bounds of $L_R(P;U)$ in terms of 'useful' R-norm information measure given by (2).

Theorem 1. If $l_i, i=1,2,3,\ldots,n$ are the lengths of codeword's x_i satisfying (4), then

$$H_R(P;U) \leq L_R(P;U) \tag{11}$$

under the condition

$$\sum_{i=1}^{n} u_i D^{-l_i} \leq \sum_{i=1}^{n} u_i p_i. \tag{12}$$

Proof: By Holder's inequality we have

$$\left(\sum_{i=1}^{n} x_i^{\,p}\right)^{1/p}\left(\sum_{i=1}^{n} y_i^{\,q}\right)^{1/q} \le \sum_{i=1}^{n} x_i y_i\ , \tag{13}$$

where $x_i, y_i \ge 0$ for each i and $\frac{1}{p}+\frac{1}{q}=1$,

$$x_i = \left(\frac{u_i p_i}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{R}{R-1}} D^{-l_i},\quad y_i = \left(\frac{u_i p_i^R}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{1-R}}$$

Putting,

$p=\frac{R-1}{R}$ and $q=1-R$ in (13) we have

$$\left[\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} u_i p_i}\right]^{\frac{R}{R-1}}\left[\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i}\right]^{\frac{1}{1-R}} \le \frac{\sum_{i=1}^{n} u_i D^{-l_i}}{\sum_{i=1}^{n} u_i p_i} \le 1$$

It implies that

$$\left[\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i}\right]^{\frac{1}{1-R}} \le \left[\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} u_i p_i}\right]^{\frac{R}{1-R}} \tag{14}$$

Case 1. Let $0<R<1$. Raising power $\frac{1-R}{R}>0$ both sides of (14) we have

$$\left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}} \le \frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} u_i p_i}$$

Subtracting both sides from 1, we get

$$1-\left(\frac{\sum_{i=1}^{n} u_i p_i^{\,R}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}} \ge 1-\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} u_i p_i} \tag{15}$$

Multiplying (15) by $\frac{R}{R-1}<0$ throughout we get

$$\frac{R}{R-1}\left[1-\left(\frac{\sum_{i=1}^{n} u_i p_i^{\,R}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}}\right] \le \frac{R}{R-1}\left[1-\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} u_i p_i}\right]$$

$$H_R(P;U) \le L_R(P;U) \tag{16}$$

Case 2. Let $R>1$. Raising power $\frac{1-R}{R}<0$ both sides of (14), we have

$$\left(\frac{\sum_{i=1}^{n} u_i p_i^{\,R}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}} \ge \frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} u_i p_i}$$

or

$$1-\left(\frac{\sum_{i=1}^{n} u_i p_i^{\,R}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}} \le 1-\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} u_i p_i} \tag{17}$$

Multiplying (17) by $\frac{R}{R-1}>0$ throughout we get:

$$\frac{R}{R-1}\left[1-\left(\frac{\sum_{i=1}^{n} u_i p_i^{\,R}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}}\right] \le \frac{R}{R-1}\left[1-\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R}{R-1}\right)}}{\sum_{i=1}^{n} u_i p_i}\right]$$

$$H_R(P;U) \le L_R(P;U) \tag{18}$$

Hence theorem 1 is proved in both cases.

In (1) equality holds if and only if

$$D^{-l_i} = \frac{p_i^R}{\sum_{i=1}^{n} u_i p_i^R \Big/ \sum_{i=1}^{n} u_i p_i}, \qquad R>0(\neq 1)$$

or

$$l_i = -\log_D p_i^R + \log_D \left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right)$$

or

$$\log_D p_l^{-R} \left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right) \le l_i < \log_D p_i^{-R} \left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right) + 1 \tag{19}$$

It implies

$$p_i^{-R} \left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right) \le D^{l_i} < D p_i^{-R} \left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right) \tag{20}$$

In the next theorem, we obtain an upper bound on $L_R(P;U)$ in term of $H_R(P;U)$.

Theorem 2. Let $l_1, l_2, \ldots, l_n$ be the code words lengths satisfying (20).Then following inequality holds:

$$L_R(P;U) < D^{\frac{1-R}{R}} H_R(P;U) + \frac{R}{R-1}\left(1 - D^{\frac{1-R}{R}}\right). \tag{21}$$

Proof: From the right hand inequality of (20), we have

$$D^{l_i} < D p_i^{-R} \left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right) \tag{22}$$

Here two cases arise:

Case1. Let $0<R<1$.Raising both sides of (22) to the power $\frac{1-R}{R} > 0$, we get

$$D^{-l_i\left(\frac{R-1}{R}\right)} < D^{\frac{1-R}{R}} p_i^{R-1} \left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1-R}{R}} \tag{23}$$

Multiplying both sides of (23) by $\frac{u_i p_i}{\sum u_i p_i}$ and summing over i, we have

$$\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} u_i p_i} < D^{\frac{1-R}{R}} \left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right) \left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1-R}{R}}$$

or

$$\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} u_i p_i} < D^{\frac{1-R}{R}} \left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{R}}$$

Subtracting both sides from 1 and multiplying by $\frac{R}{R-1} < 0$, we have

$$L_R(P;U) < D^{\frac{1-R}{R}} H_R(P;U) + \frac{R}{R-1}\left(1 - D^{\frac{1-R}{R}}\right) \tag{24}$$

Similarly we can prove that (24) holds when $R>1$. Hence theorem 2 is proved.

Thus from (11) and (21)

$$H_R(P;U) \le L_R(P;U) < D^{\frac{1-R}{R}} H_R(P;U) + \frac{R}{R-1}\left(1 - D^{\frac{1-R}{R}}\right). \tag{25}$$

Which is Shannon's Coding theorem for noiseless channel on 'useful' R-norm information measure.

Lower and Upper Bounds of Generalized Cost Measure of Degree β

Let $P=\{p_1, p_2, \dots, p_n, 0\le p_i<1, \sum p_i=1\}$ be replaced by β degree probability distribution given by $P^\beta=\{p_1^\beta, p_2^\beta, \dots, p_n^\beta, 0\le p_i^\beta<1, \sum p_i^\beta\le 1\}$, where $\beta\ge 1$. Then (9) becomes

$$L_R^\beta(P;U)=\frac{R}{R-1}\left[1-\frac{\sum_{i=1}^n u_i p_i^\beta D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^n u_i p_i^\beta}\right]. \qquad (26)$$

In this section we study the lower and upper bounds for $L_R^\beta(P;U)$ in term of the generalized 'useful' R-norm information measure of degree β as given below:

$$H_R^\beta(P;U)=\frac{R}{R-1}\left[1-\left(\frac{\sum_{i=1}^n u_i p_i^{R\beta}}{\sum_{i=1}^n u_i p_i}\right)^{\frac{1}{R}}\right], \quad R>0(\ne 1), \beta\ge 1, p_i\ge 0\ \forall i=1,2,\dots,n \qquad (27)$$

under the condition

$$\sum_{i=1}^n u_i D^{-l_i}\le \sum_{i=1}^n u_i p_i^\beta . \qquad (28)$$

In the next theorem we find the lower bound for $L_R^\beta(P;U)$ in term of 'useful' R-norm information measure of degree β.

Theorem 3. Let $\{u_i\}_{i=1}^n, \{p_i\}_{i=1}^n, \{l_i\}_{i=1}^n$, satisfy the inequality (27), then

$$L_R^\beta(P;U)\ge H_R^\beta(P;U), \quad R>0(\ne 1), \quad \beta\ge 1, \qquad (29)$$

where $L_R^\beta(P;U)$ and $H_R^\beta(P;U)$ are given by (26) and (27) respectively.

Proof: By Holder's inequality we have

$$\left(\sum_{i=1}^n x_i^{\,p}\right)^{1/p}\left(\sum_{i=1}^n y_i^{\,q}\right)^{1/q}\le \sum_{i=1}^n x_i y_i , \qquad (30)$$

where $x_i, y_i\ge 0$ for each i

and

$$\frac{1}{p}+\frac{1}{q}=1;\ p(\ne 0)<1, q<0$$

or

$$q(\ne 0)<1, p<0; x_i, y_i>0 \text{ for each } i$$

Putting

$$x_i=\left(\frac{u_i p_i^\beta}{\sum_{i=1}^n u_i p_i^\beta}\right)^{\frac{R}{R-1}} D^{-l_i}, \quad y_i=\left(\frac{u_i p_i^{R\beta}}{\sum_{i=1}^n u_i p_i^\beta}\right)^{\frac{1}{1-R}},$$

$$p=\frac{R-1}{R} \text{ and } q=1-R$$ in (30)

we have

$$\left[\frac{\sum_{i=1}^n u_i p_i^\beta D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^n u_i p_i^\beta}\right]^{\frac{R}{R-1}}\left[\frac{\sum_{i=1}^n u_i p_i^{R\beta}}{\sum_{i=1}^n u_i p_i^\beta}\right]^{\frac{1}{1-R}}\le \frac{\sum_{i=1}^n u_i D^{-l_i}}{\sum_{i=1}^n u_i p_i^\beta}\le 1$$

It implies that

$$\left[\frac{\sum_{i=1}^n u_i p_i^{R\beta}}{\sum_{i=1}^n u_i p_i^\beta}\right]^{\frac{1}{1-R}}\le\left[\frac{\sum_{i=1}^n u_i p_i^\beta D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^n u_i p_i^\beta}\right]^{\frac{R}{1-R}} \qquad (31)$$

Case 1. Let $0<R<1$. Raising power $\frac{1-R}{R}>0$ both sides of (31) we have

$$\left(\frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right)^{\frac{1}{R}} \le \frac{\sum_{i=1}^{n} u_i p_i^{\beta} D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} u_i p_i^{\beta}}$$

Subtracting both sides from 1, we get

$$1-\left(\frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right)^{\frac{1}{R}} \ge 1-\frac{\sum_{i=1}^{n} u_i p_i^{\beta} D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} u_i p_i^{\beta}} \tag{32}$$

Multiplying (32) by $\frac{R}{R-1}<0$ throughout we get

$$\frac{R}{R-1}\left[1-\left(\frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right)^{\frac{1}{R}}\right] \le \frac{R}{R-1}\left[1-\frac{\sum_{i=1}^{n} u_i p_i^{\beta} D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right]$$

$$H_R^{\beta}(P;U) \le L_R^{\beta}(P;U) \tag{33}$$

Case 2. Let $R>1$. Raising power $\frac{1-R}{R}<0$ both sides of (31) we have

$$\left(\frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right)^{\frac{1}{R}} \ge \frac{\sum_{i=1}^{n} u_i p_i^{\beta} D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} u_i p_i^{\beta}}$$

$$1-\left(\frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right)^{\frac{1}{R}} \le 1-\frac{\sum_{i=1}^{n} u_i p_i^{\beta} D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} u_i p_i^{\beta}} \tag{34}$$

Multiplying both sides by $\frac{R}{R-1}>0$, we get

$$\frac{R}{R-1}\left[1-\left(\frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p^{\beta}}\right)^{\frac{1}{R}}\right] \le \frac{R}{R-1}\left[1-\frac{\sum_{i=1}^{n} u_i p_i^{\beta} D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right]$$

$$H_R(P;U) \le L_R(P;U) \tag{35}$$

Hence theorem 3 is proved in both cases.

1. In (3) equality holds if and only if

$$D^{-l_i} = \frac{p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{R\beta} \Big/ \sum_{i=1}^{n} u_i p_i^{\beta}}, R>0(\ne 1), \beta>0$$

or

$$l_i = -\log_D p_i^{R\beta} + \log_D \left(\frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right)$$

or

$$\log_D p_i^{-R\beta}\left(\frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right) \le l_i < \log_D p_i^{-R\beta}\left(\frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right)+1 \tag{36}$$

It implies

$$p_i^{-R\beta}\left(\frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right) \le D^{l_i} < D p_i^{-R\beta}\left(\frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=11}^{n} u_i p_i^{\beta}}\right) \tag{37}$$

In the next theorem, we obtain an upper bound on $L_R^{\beta}(P;U)$ in term of $H_R^{\beta}(P;U)$.

Theorem 4. Let $l_1, l_2, \ldots, l_n$ be the code words lengths satisfying (37).Then following inequality holds:

$$L_R^{\beta}(P;U) < D^{\frac{1-R}{R}} H_R^{\beta}(P;U) + \frac{R}{R-1}\left(1-D^{\frac{1-R}{R}}\right) \tag{38}$$

Proof: From the right hand inequality of (37), we have

$$D^{l_i} < D p_i^{-R\beta}\left(\frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right) \tag{39}$$

Here two cases arise:

Case 1. Let $0 < R < 1.$ Raising both sides of (39) to the power $\frac{1-R}{R} > 0,$ we get

$$D^{-l_i\left(\frac{R-1}{R}\right)} \le D^{\left(\frac{1-R}{R}\right)} p_i^{\beta\left(\frac{R-1}{R}\right)}\left(\frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right)^{\left(\frac{1-R}{R}\right)} \tag{40}$$

Multiplying by $\frac{u_i p_i^{\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta}}$ and summing over i, we have

$$\left(\frac{\sum_{i=1}^{n} u_i p_i^{\beta} D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right) < D^{\left(\frac{1-R}{R}\right)}\left(\frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right)^{\frac{1}{R}}$$

Subtracting both sides from1 and multiplying by $\frac{R}{R-1} < 0$, we have

$$L_R^{\beta}(P;U) < D^{\frac{1-R}{R}} H_R^{\beta}(P;U) + \frac{R}{R-1}\left(1 - D^{\frac{1-R}{R}}\right) \tag{41}$$

Similarly in case 2, (38) holds when $R > 1.$ Hence theorem 4 is proved.

Thus (29) and (38) together give

$$H_R^{\beta}(P;U) \le L_R^{\beta}(P;U) < D^{\frac{1-R}{R}} H_R^{\beta}(P;U) + \frac{R}{R-1}\left(1 - D^{\frac{1-R}{R}}\right) \tag{42}$$

which is Shannon's coding theorem for noiseless channel on 'useful' R-norm information measure of degree β.

Particular cases:

(i) When $\beta = 1$, (29) reduces to (11).

(ii) When $\beta = 1$, (38) reduces to (21).

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