Direct RESEARCH JOURNALS

Direct Research Journal of Engineering and Information Technology
Vol. 7(2), pp. 55-57, June 2020
ISSN 2354-4155
DOI: https://doi.org/10.26765/DRJEIT21060549
Article Number: DRJEIT21060549
Copyright © 2020
Author(s) retain the copyright of this article
http://directresearchpublisher.org/drjeit/

Full Length Research Paper

A Semi-analytical Solution of the Black-Scholes Pricing Model for European Call Option

Durojaye, M. O.*, and Kazeem J. A.

Department of Mathematics, University of Abuja, Nigeria.
*Corresponding Author E-mail: mayojaye@yahoo.com

Received 3 June 2020; Accepted 24 June, 2020

ABSTRACT: This paper considers approximate solution of Black-Scholes pricing model for European call option using a semi-analytical method. In this approach, the second-order parabolic partial differential equation (PDE) is transformed into a system of ordinary differential equations (ODEs) and solved. It is observed that the approximate values obtained are in good agreement with previous results with less computational and programming efforts.

Keywords: Black-Scholes Model, European Call Option, Method of Lines (MOL)

INTRODUCTION

Black-Scholes model, over years, has been a major interest in mathematics of finance. It is a second-order parabolic partial differential equation (PDE) as well as a variation of a famous equation in physics that models the transfer of heat (Anwar and Andallah, 2018). The model was first published in 1973 by Black and Scholes (1973). It assumes the existence of perfect capital markets and the security prices are log normally distributed. In addition, there is assumption that trading in all securities is continuous with the distribution of the rates of return being stationary (Allahviranloo and Behzadi, (2013). In recent years, a number of analytical and numerical techniques have been used for solving Black-Scholes equation.

In 2005, Jódar et al. (2005) used Mellin transformation where neither the variable transformation nor solving diffusion equation was required, to obtain the solution of the model. Jodar et al., (2006) in 2006 solved the modified Black-Scholes equation pricing option with discrete dividend using a delta-defining sequence of generalized Dirac-Delta function and applied the Mellin transformation to obtain an integral formula, after which they approximated the solution using numerical quadrature approximation. In 2009, Bohner and Zheng (2009) also worked on the analytical solution of the model.

Motivated by various analytical and numerical approaches in solving Black-Scholes equation, in this paper, we discuss a semi-analytical method called Method of Lines (MOL) in solving the Black-Scholes initial (final) boundary value problem (BS-IFBVP) for European call option.

BLACK-SCHOLES MODEL

Black-Scholes model is a mathematical model for the evaluation of an option price. It is a function of variables: Volatility, Stock Price, Striking Price, Time to Expiration, Dividends to be paid and the risk-free interest rate for

pricing European put or call options on Stocks. While developing the model in 1973, the following assumptions were considered by (Black and Scholes, 1973; Shinde and Takale, 2012):

(i) The underlying asset is a stock that does not pay any dividend during the life of the contract.
(ii) The underlying asset price is driven by a geometric Brownian motion with constant volatility
(iii) Market direction in unpredictable hence it follows 'Random Walk'.
(iv) There are no transaction costs (taxes or brokerage fees or commissions).
(v) Interest rates remain constant.
(vi) The market operates uninterruptedly; there is no calendar but trading days i.e. option is exercised upon expiration.

The Black-Scholes equation which is a second order parabolic partial differential equation is given by (Anwar and Andallah, 2018):

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 + \frac{\partial^2 V}{\partial S^2} - rV = 0 \quad (1)$$

where $V(S,t)$-the price of an option, $S\epsilon[0,\infty)$-the current option price of the stock, σ-the volatility of the underlying asset, r-the annual risk-free interest rate compounded continuously, $t\epsilon[0,T]$-contract lifespan in years with T being the expiry time.

Considering European call option, we denote $V(S,t)$ in eq. (1) by $C(S,t)$ and define the initial and boundary conditions. Specifically, we must consider final and boundary condition for the case of European Call option. The value at final time $t = T$ can be obtained using the definition of call option "an option that grants its holder the right to buy the underlying asset at a strike price (K) at some moment in the future".

(i) If $S > K$ (Predetermined price of an underlying asset) at expiry, $t = T$, it makes financial sense to exercise the call option, handing over an amount K , to obtain an asset worth S. The profit from such a transaction is then, $S - K$.
(i) If $S < K$ at expiry, $t = T$, it does not make financial sense to exercise the call option as there will be a loss of, $S - K$ and the option expires worthless.
Thus, the value of a call option at expiry called 'the final condition' can be written as:

$$C(S,T) = Max(S - K, 0) \quad (2)$$

At time, $t = 0$, the value of the option is known to be the payoff given by:

$$C(S,0) = Max(S - K, 0) \quad (3)$$

Eq. (3) is called the initial condition
In obtaining the boundary conditions, the value of $C(S,t)$ is considered at $S = 0$ and $S \to \infty$.
If $S = 0$, then, from eq. (1), we have that the payoff must be zero and consequently,

$$C(0,t) = 0 \quad (4)$$

If $S \to \infty$, it seems more likely that the option will be exercised and the payoff will be $S - K$. Also, the strike price becomes insignificant as $S \to \infty$, then, the value of the option is equivalent to

$$C(S,t) = S - Ke^{-r(T-t)} \text{ as } S \to \infty \quad (5)$$

Now, by replacing $V(S,t)$ in eq. (1) by $C(S,t)$ and using equations (2) – (5), we obtain the following Black-Scholes initial (final) boundary value problem (BS-IFBVP) for European call option:

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 + \frac{\partial^2 C}{\partial S^2} - rC = 0,$$
$$0 \le S < \infty,\ 0 \le t \le T \quad (6)$$

Initial condition: $C(S,0) = Max(S - K, 0)$,
$0 \le S < \infty$ (7)

Final condition: $C(S,T) = Max(S - K, 0)$,
$0 \le S < \infty$ (8)

Boundary equations: $C(0,t) = 0$, $0 \le t \le T$ and $C(S,t) = S - Ke^{-r(T-t)}$ as $S \to \infty$ (9)

METHOD OF LINES (MOL)

The basic idea of the MOL is to replace the spatial (boundary value) derivatives in the PDE with algebraic approximations. Once this is done, only the initial value variable, typically time in a physical problem, remains. Then, with only one remaining independent variable, we have a system of ODEs that approximates the original PDE. Any suitable integration algorithm for initial value ODEs can now be used to compute an approximate numerical solution to the PDE (Biazar and Nomidi, 2013; Schiesser, 1991; Knapp, 2008).

Applying the Method of Lines in equation (6) - (9) by discretizing in space variable S and leaving the variable t continuous, we have:

$$\left(\frac{dC}{dt}\right)_i = rC_i - rS_i\frac{(C_{i+1} - C_{i-1})}{2\Delta S} - \frac{1}{2}\sigma^2 S_i^2\frac{C_{i+1} - 2C_i + C_{i-1}}{(\Delta S)^2} \quad (10)$$

$$C(S_i,0) = C(S_i,T) = Max(S_i - K, 0) = f(S_i,K), \quad (11)$$

$$C(0,t) = C_o(0,t) = 0 \quad (12)$$

$$C(\infty,t) \cong C(N+1,t) = S_i - Ke^{-r(T-t)} = g(S_i,K,r,t,T), \quad i = 1,2,3 \ldots N \quad (13)$$

Simplifying eq. (10), we have:

$$\left(\frac{dC}{dt}\right)_i = \alpha_1 C_{i-1} + \alpha_2 C_i + \alpha_3 C_{i+1} \quad (14)$$

where

$$\alpha_1 = \frac{rS_i}{2\Delta S} - \frac{1}{2}\sigma^2\frac{S_i^2}{(\Delta S)^2}, \quad \alpha_2 = r + \frac{\sigma^2 S_i^2}{\Delta S}, \quad \alpha_3 = -\left(\frac{rS_i}{2\Delta S} + \frac{1}{2}\sigma^2\frac{S_i^2}{(\Delta S)^2}\right) \quad (15)$$

$$i = 1,2,3 \ldots N, \quad \Delta S = \frac{S}{N}$$

In Matrix form, eq. (14) with boundary conditions in eq.'s (12) – (13), for $i = 1,2,3 \ldots N$, can be written as:

$$\begin{bmatrix} \dot{C}_1 \\ \dot{C}_2 \\ \vdots \\ \dot{C}_{N-1} \\ \dot{C}_N \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & \alpha_1 & \alpha_2 & \alpha_3 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} \begin{bmatrix} 0 \\ C_1 \\ C_2 \\ \vdots \\ C_N \\ g(S_i,K,r,t,T) \end{bmatrix} \quad (16)$$

where the coefficients α_1, α_2, and α_3 are given by eq. (15) and $\dot{C}_i = \left(\frac{dC}{dt}\right)_i$

RESULTS AND DISCUSSION

In this paper, a semi-analytical method called Method of lines (MOL) had been used to find the approximate solution to Black-Scholes initial (final) boundary value problem (BS-IFBVP) for European call option. For computational outputs, the MATLAB code was used for the following values of parameters: $0 \leq S \leq 20$, $T = 0.25$, $K = 10$, $r = 0.1$, $\sigma = 0.4$. The results obtained here are compared with exact values and some other existing approximate values: semi-implicit method (Dura and Mosneagu, 2010), explicit method (Anwar and Andallah, 2018), hybrid method Olaiya et al., 2019), they were found to be in agreement, as shown in (Table 1).

Conclusion

It has been shown in this paper that Method of Lines (MOL) is also an effective and efficient numerical method with less computational and programming efforts in obtaining the numerical solution of Black-Scholes equations for European call option.

REFERENCES

Allahviranloo T, Behzadi S (2008). "The use of iterative methods for solving Black-Scholes equation" Int. J. Industrial Mathematics (ISSN 2008-5621) Vol. 5, No. 1, 2013 Article ID IJIM-00329, 11 pages

Anwar MN, Andallah L S (2018): "A Study on Numerical Solution of Black-Scholes Model" Journal of Mathematical Finance, 8, 372-381. http://www.scirp.org/journal/jmf. OI: 10.4236/jmf.2018.82024 May 18, 2018 372 Journal of Mathematical Finance

Biazar J, Nomidi N (2013). Solution of the Emdem-Fowler equation by the method of lines. Journal of Nature Science and Sustainable Technology. 2013;7 (2):45-55.

Black F, Scholes M (1973): "The Pricing of Options and Corporate Liabilities. Journal of Political Economy" 81, 637-654. https://doi.org/10.1086/260062

Bohner M, Zheng Y (2009). "On analytical solution of the Black-Scholes equation", Appl. Math. Lett 22:309-313.

Dura G, Mosneagu AM (2010). Numerical Approximation of Black-Scholes Equation. An. Stiint. Univ."Al. I. Cuza" Iasi. Mat, 56, 39-64.

Jodar L, Company LR, Gonzalez AL (2006). Numerical solution of modified Black-Scholes equation pricing stock options with discrete dividend. Mathematical and Computer Modelling, 44, 2006.

Jódar LP, Sevilla-Peris JC, Cortés RS (2005). A new direct method for solving the Black-Scholes equation. *Applied Mathematics Letters*, 18, 2005.

Knapp R (2008). A method of lines framework in Mathematica. Journal of Numerical Analysis, Industrial and Applied Mathematics. 3 (1):43-59.

Olaiya OO, Oduwole HK, Odeyemi JK (2019)."Numerical solution of black –Scholes partial differential equation using direct solution of second-order ordinary differential equation with two-step hybrid block method of order seven" *Science World Journal Vol. 14(No 2) 2019. www.scienceworldjournal.org ISSN 1597-6343 Published by Faculty of Science, Kaduna State University.*

Schiesser WE (1991): The numerical method of lines- integration of partial differential equations. Academic Press, San Diego.

Shinde AS, Takale KS (2012). "Study of Black-Scholes Model and its Application" International Conference on Modelling, Optimization and Computing (ICMOC-2012). Procedia Engineering 38(2012):270-279.